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Advanced Linear Algebra (MA 409)
Problem Sheet - 6

Linear Transformations, Null Spaces, and Ranges

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1. Label the following statements as true or false. In each part, V and W are finite-dimensional vector spaces (over F), and T is a function from V to W.
   (a) If T is linear, then T preserves sums and scalar products.
   (b) If $T(x + y) = T(x) + T(y)$, then T is linear.
   (c) T is one-to-one if and only if the only vector x such that $T(x) = 0$ is $x = 0$.
   (d) If T is linear, then $T(0_V) = 0_W$.
   (e) If T is linear, then $\text{nullity}(T) + \text{rank}(T) = \dim(W)$.
   (f) If T is linear, then T carries linearly independent subsets of V onto linearly independent subsets of W.
   (g) If $T, U : V \to W$ are both linear and agree on a basis for V, then $T = U$.
   (h) Given x_1, $x_2 \in V$ and y_1, $y_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.

2. In the following exercises, prove that T is a linear transformation, and find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems to determine whether T is one-to-one or onto.
   (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.
   (b) $T : M_{2\times 3}(F) \to M_{2\times 2}(F)$ defined by

$$T \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}.$$

   (c) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$.
   (d) $T : M_{n\times n}(F) \to F$ defined by $T(A) = tr(A)$.

3. In this exercise, $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a function. For each of the following parts, state why T is not linear.
   (a) $T(a_1, a_2) = (1, a_2)$
   (b) $T(a_1, a_2) = (a_1, a_1^2)$
   (c) $T(a_1, a_2) = (\sin a_1, 0)$
   (d) $T(a_1, a_2) = (|a_1|, a_2)$
   (e) $T(a_1, a_2) = (a_1 + 1, a_2)$

4. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, $T(1,0) = (1,4)$, and $T(1,1) = (2,5)$. What is $T(2,3)$? Is T one-to-one?

5. Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?

6. Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$?

7. Let V and W be vector spaces, let $T : V \to W$ be linear, and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of $R(T)$. Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \ldots, k$, then S is linearly independent.

8. Let V and W be vector spaces and $T : V \to W$ be linear.

   (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

   (b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

   (c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.

9. Define

$$T : P(\mathbb{R}) \to P(\mathbb{R}) \text{ by } T(f(x)) = \int_0^x f(t)dt.$$

   Prove that T linear and one-to-one, but not onto.

10. Let $T : P(\mathbb{R}) \to P(\mathbb{R})$ be defined by $T(f(x)) = f'(x)$. Recall that T is linear. Prove that T is onto, but not one-to-one.

11. Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.

    (a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.

    (b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

12. Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.

13. Give an example of distinct linear transformations T and U such that $N(T) = N(U)$ and $R(T) = R(U)$.

14. Let V and W be vector spaces with subspaces V_1 and W_1, respectively. If $T : V \to W$ is linear, prove that $T(V_1)$ is a subspace of W and that $\{x \in V : T(x) \in W_1\}$ is a subspace of V.

15. Let V be the vector space of sequences. Define the functions $T, U : V \to V$ by

$$T(a_1, a_2, \ldots) = (a_2, a_3, \ldots) \text{ and } U(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots).$$

    T and U are called the **left shift** and **right shift** operators on V, respectively.

    (a) Prove that T and U are linear.

    (b) Prove that T is onto, but not one-to-one.

(c) Prove that U is one-to-one, but not onto.

16. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be linear. Show that there exist scalars a, b, and c such that $T(x, y, z) = ax + by + cz$ for all $(x, y, z) \in \mathbb{R}^3$. Can you generalize this result for $T : F^n \to F$? State and prove an analogous result for $T : F^n \to F^m$.

17. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be linear. Describe geometrically the possibilities for the null space of T.

18. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$. Include figures for each of the following parts.

   (a) Find a formula for $T(a, b)$, where T represents the projection on the y-axis along the x-axis.

   (b) Find a formula for $T(a, b)$, where T represents the projection on the y-axis along the line $L = \{(s, s) : s \in R\}$.

19. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$.

   (a) If $T(a, b, c) = (a, b, 0)$, show that T is the projection on the xy-plane along the z-axis.

   (b) Find a formula for $T(a, b, c)$, where T represents the projection on the z-axis along the xy-plane.

   (c) If $T(a, b, c) = (a - c, b, 0)$, show that T is the projection on the xy-plane along the line $L = \{(a, 0, a) : a \in \mathbb{R}\}$.

20. Using the notation in the definition above, assume that $T : V \to V$ is the projection on W_1 along W_2.

   (a) Prove that T is linear and $W_1 = \{x \in V : T(x) = x\}$.

   (b) Prove that $W_1 = R(T)$ and $W_2 = N(T)$.

   (c) Describe T if $W_1 = V$.

   (d) Describe T if W_1 is the zero subspace.

21. Suppose that W is a subspace of a finite-dimensional vector space V.

   (a) Prove that there exists a subspace W' and a function $T : V \to V$ such that T is a projection on W along W'.

   (b) Give an example of a subspace W of a vector space V such that there are two projections on W along two (distinct) subspaces.

22. Prove that the subspaces $\{0\}, V, R(T)$, and $N(T)$ are all T-invariant.

23. If W is T-invariant, prove that T_W is linear.

24. Suppose that T is the projection on W along some subspace W'. Prove that W is T-invariant and that $T_W = l_W$.

25. Suppose that $V = R(T) \oplus W$ and W is T-invariant.

   (a) Prove that $W \subseteq N(T)$.

   (b) Show that if V is finite-dimensional, then $W = N(T)$.

   (c) Show by example that the conclusion of (b) is not necessarily true if V is not finite-dimensional.

26. Suppose that W is T-invariant. Prove that $N(T_W) = N(T) \cap W$ and $R(T_W) = T(W)$.

27. Let V and W be vector spaces, and let $T : V \to W$ be linear. If β is a basis for V, then prove that

$$R(T) = \text{span}(\{T(v) : v \in \beta\}).$$

28. Let V and W be vector spaces over a common field, and let β be a basis for V. Then for any function $f : \beta \to W$ there exists exactly one linear transformation $T : V \to W$ such that $T(x) = f(x)$ for all $x \in \beta$.

29. Let V be a finite-dimensional vector space and $T : V \to V$ be linear.

   (a) Suppose that $V = R(T) + N(T)$. Prove that $V = R(T) \oplus N(T)$.

   (b) Suppose that $R(T) \cap N(T) = \{0\}$. Prove that $V = R(T) \oplus N(T)$.

30. Let V be the vector space of sequences. Define $T : V \to V$ by

$$T(a_1, a_2, \ldots) = (a_2, a_3, \ldots).$$

   (a) Prove that $V = R(T) + N(T)$, but V is not a direct sum of these two spaces. Thus the result of Exercise 29(a) above cannot be proved without assuming that V is finite-dimensional.

   (b) Find a linear operator T_1 on V such that $R(T_1) \cap N(T_1) = \{0\}$ but V is not a direct sum of $R(T_1)$ and $N(T_1)$. Conclude that V being finite-dimensional is also essential in Exercise 29(b).

31. A function $T : V \to W$ between vector spaces V and W is called **additive** if $T(x + y) = T(x) + T(y)$ for all $x, y \in V$. Prove that if V and W are vector spaces over the field of rational numbers, then any additive function from V into W is a linear transformation.

32. Let $T : \mathbb{C} \to \mathbb{C}$ be the function defined by $T(z) = \bar{z}$. Prove that T is additive (as defined in the above exercise) but not linear.

33. Prove that there is an additive function $T : \mathbb{R} \to \mathbb{R}$ that is not linear.

    [Hint : Let V be the set of real numbers regarded as a vector space over the field of rational numbers. As every vector space has a basis, V has a basis β. Let x and y be two distinct vectors in β, and define $f : \beta \to V$ by $f(x) = y$, $f(y) = x$, and $f(z) = z$ otherwise. Hence there exists a linear transformation $T : V \to V$ such that $T(u) = f(u)$ for all $u \in \beta$. Then T is additive, but for $c = y/x$, $T(cx) \neq cT(x)$.]

34. Let V be a vector space and W be a subspace of V. Define the mapping $\eta : V \to V/W$ by $\eta(v) = v + W$ for $v \in V$.

   (a) Prove that η is a linear transformation from V onto V/W and that $N(\eta) = W$.

   (b) Suppose that V is finite-dimensional. Use (a) and the dimension theorem to derive a formula relating $\dim(V)$, $\dim(W)$, and $\dim(V/W)$.

   (c) Read the proof of the dimension theorem. Compare the method of solving (b) with the method of deriving the same result as outlined below :

   Let W be a subspace of a finite-dimensional vector space V, and consider the basis $\{u_1, u_2, \ldots, u_k\}$ for W. Let $\{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ be an extension of this basis to a basis for V. Then $\{u_{k+1} + W, u_{k+2} + W, \ldots, u_n + W\}$ is a basis for V/W and

$$\dim(W) + \dim(V/W) = \dim(V).$$